

MEASURING POSITION AND MOMENTUM TOGETHER

PAUL BUSCH, JUKKA KIUKAS, AND PEKKA LAHTI

ABSTRACT. We describe an operational scheme for determining both the position and momentum distributions in a large class of quantum states, together with an experimental implementation.

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The basic understanding of quantum mechanics asserts that due to their noncommutativity, position and momentum of a quantum object cannot be measured together in any state. It is an equally well known fact that these observables can be *approximately* measured together if the measurement accuracies fulfill an appropriate trade-off relation. Much effort has gone into obtaining rigorous formulations of these facts; for a recent overview see, for instance, [1].

In this paper we wish to point out that there are measurement schemes which allow one to determine both the position and the momentum distributions from the statistics of a *single* measurement: in this sense the position and momentum of a quantum object can, after all, be measured together.

Technically, this observation builds on the well-known properties of some covariant phase space observables, such as the Husimi Q-function. The measurement in question can then be implemented via eight-port homodyne detection. Therefore, the idea is completely realistic.

The structure of covariant phase space observables is well-known. If S is a positive operator of trace one (acting on the Hilbert space \mathcal{H} of the quantum object), and W_{qp} , $(q, p) \in \mathbb{R}^2$, are the Weyl operators associated with the position and momentum operators Q and P , then the operators

$$G^S(Z) = \frac{1}{2\pi} \int_Z W_{qp} S W_{qp}^* dq dp,$$

(with the integral being defined in the sense of the weak operator topology) constitute a covariant phase space observable G^S (defined on the Borel subsets of the phase space \mathbb{R}^2). Conversely, any covariant phase space observable is of that form for some S [2]. If the generating operator S is the vacuum state $|0\rangle$, then the probability distribution $Z \mapsto G_\rho^S(Z) \equiv \text{Tr}[\rho G^S(Z)] = \text{Tr}[\rho G^{(0)}(Z)]$ is the Husimi distribution of the state (density operator) ρ .

We are going to show that for a certain family of operators S the phase space observable G^S entails complete information about the position and momentum observables: that is, the spectral measures of Q and P can be determined from G^S . This is done by using the moment operators $L(x^k, G^S)$ and $L(y^k, G^S)$ of G^S , which are defined as follows (using the general theory of operator integrals):

$$\begin{aligned} L(x^k, G^S) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} q^k W_{qp} S W_{qp}^* dq dp, \\ L(y^k, G^S) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} p^k W_{qp} S W_{qp}^* dq dp. \end{aligned}$$

In general, it is very difficult to determine the domains of these (unbounded) operators. However, the following is known [7, Theorem 4] (for a similar result, see [8]): for any $k \in \mathbb{N}$, if (and only if) the operator $Q^k \sqrt{S}$ is a Hilbert-Schmidt operator, that is, $\text{Tr}[SQ^{2k}] < \infty$, then

$$(1) \quad L(x^k, G^S) = \sum_{l=0}^k s_{kl}^Q Q^l, \quad \text{with } s_{kl}^Q = \binom{k}{l} (-1)^{k-l} \text{Tr}[Q^{k-l} S],$$

and, similarly, if (and only if) the operator $P^k\sqrt{S}$ is a Hilbert-Schmidt operator, then

$$(2) \quad L(y^k, G^S) = \sum_{l=0}^k s_{kl}^P P^l, \text{ with } s_{kl}^P = \binom{k}{l} (-1)^{k-l} \text{Tr}[P^{k-l} S].$$

For any vector state φ , the numbers $\langle \varphi | L(x^k, G^S) \varphi \rangle$ are the k^{th} moments of the marginal probability measure $X \mapsto \langle \varphi | G^S(X \times \mathbb{R}) \varphi \rangle$, and similarly for the numbers $\langle \varphi | L(y^k, G^S) \varphi \rangle$. From Eq (1) one can easily determine any Q^k in terms of the (operationally well defined) moment operators $L(x^l, G^S)$, $l = 0, \dots, k$, and, similarly, each P^k in terms of the (operationally well defined) moment operators $L(y^l, G^S)$, $l = 0, \dots, k$, from (2). More explicitly, consider the simple recursive transformation

$$(3) \quad \beta_k = \alpha_k - \sum_{l=0}^{k-1} s_{kl} \beta_l, \quad k \in \mathbb{N}.$$

Putting $\alpha_k = \langle \varphi | L(x^k, G^S) \varphi \rangle$ and $s_{kl} = s_{kl}^Q$ yields $\beta_k = \langle \varphi | Q^k \varphi \rangle$. (Note that $s_{kk}^Q = 1$.) Similarly, one gets the moments $\langle \varphi | P^k \varphi \rangle$ from the actual marginal moments $\langle \varphi | L(y^k, G^S) \varphi \rangle$. Therefore, the numbers $\langle \varphi | Q^k \varphi \rangle$ and $\langle \varphi | P^k \varphi \rangle$ are just as operational as the actually measured moments $\langle \varphi | L(x^k, G^S) \varphi \rangle$ and $\langle \varphi | L(y^k, G^S) \varphi \rangle$, from which they can be reconstructed via (3). But $\langle \varphi | Q^k \varphi \rangle$ and $\langle \varphi | P^k \varphi \rangle$ are the moments of the probability distributions p_φ^Q and p_φ^P of the position and momentum observables Q and P in the initial state φ . If φ is chosen to be, for example, a linear combination of Hermite functions, the measures p_φ^Q and p_φ^P are exponentially bounded, and, as such, are uniquely determined by their respective moment sequences $(\langle \varphi | Q^k \varphi \rangle)_k$ and $(\langle \varphi | P^k \varphi \rangle)_k$ [9, p. 406, Theorem 30.1]. *In this sense one is able to measure the noncommuting observables Q and P simultaneously in such a vector state φ .* Furthermore, since the linear combinations of Hermite functions are dense in $L^2(\mathbb{R})$, their associated distributions p_φ^Q and p_φ^P suffice to determine the whole position and momentum observables Q and P (as spectral measures).

The phase space observables G^S , with the generating operator S being such that $\text{Tr}[W_{qp} S] \neq 0$ for almost all $(q, p) \in \mathbb{R}^2$, are informationally complete: that is, the statistics G_ρ^S determine the state ρ of the quantum object. Given the state ρ , the distributions of all observables are then determined, including, in particular, p_ρ^Q and p_ρ^P . In this sense one might argue that an informationally complete measurement serves as a measurement of any other observable of the system. However, to our knowledge, there is no simple method to determine from the statistics G_ρ^S the statistics of another observable. An explicit reconstruction of the state typically requires *several different* measurements, as is the case with the usual tomographic methods.

In the present case we are able to construct directly and in simple operational terms (without any use of the informational completeness) the moments of the distributions p_φ^Q and p_φ^P from the statistics G_φ^S for any state φ , and these moment sequences identify, in turn, the distributions themselves in the exponentially bounded case.

It is useful to note that the moment sequence of, say, the momentum distribution does not, in general, identify that distribution, even in cases where all the moments are finite. For example, if $\varphi_\delta = (\varphi_1 + e^{i\delta} \varphi_2) / \sqrt{2}$ is a *double-slit state*, that is, φ_1, φ_2 are (smooth) wave functions supported on disjoint intervals, then the moments $\langle \varphi_\delta | P^k \varphi_\delta \rangle$ of the momentum distribution $p_{\varphi_\delta}^P$ are all finite and independent of δ although the distribution does depend on δ . This highlights the fact that the assumption of exponential boundedness is crucial in the above consideration.

We also want to point out that the exponential boundedness condition itself is operational, since it can, in principle, be verified by considering the very moment sequence that is obtained in the measurement. In fact, a probability measure with the moment sequence m_1, m_2, \dots is exponentially bounded if and only if

$$(4) \quad |m_k| \leq CR^k k!, \quad k = 1, 2, \dots,$$

for some $C, R > 0$ (see e.g. the proof of Proposition 2 of [10]).

Experimental implementations of observables G^S can be obtained in quantum optics. For example, the Husimi Q-function of a single-mode signal field can actually be measured by the eight-port homodyne detection technique, provided that the reference beam (used for the homodyne detection) is a very strong coherent field so that it can be treated classically; see e.g. [11]. In a recent paper it has been shown, *without any classicality assumptions*, that *any* covariant phase space observable G^S can be realized as a high-amplitude limit of an eight-port homodyne detection observable choosing the auxiliary parameter field to be in the conjugated state $C^{-1}SC$ [12].

In this measurement, the input signal field, in state ρ , is mixed by a beam splitter with another single-mode field, the latter being in the reference state $C^{-1}SC$. Each of the two output beams of the beam splitter are led to a homodyne detector which measures a field quadrature. One measures $\frac{1}{\sqrt{2}}(a^* + a)$ and the other one $\frac{1}{\sqrt{2}}i(b^* - b)$, where a and b are the annihilation operators for the signal and the reference field (in the Schrödinger picture). When this measurement is interpreted with respect to the signal field in the initial state ρ , the combined statistics of the homodyne detectors are given by the distribution $(X, Y) \mapsto \text{Tr}[\rho G^S(X \times Y)]$, where the Weyl operators W_{qp} are now defined in terms of the signal field quadratures $Q := \frac{1}{\sqrt{2}}(a^* + a)$ and $P := \frac{1}{\sqrt{2}}i(a^* - a)$.

By performing the measurement, one thus obtains two outcomes, which then are to be interpreted with respect to the signal beam. Notice that the outcomes for the two homodyne detectors are obtained at the same time. Consequently, upon repeating the preparation and the measurement, the statistics for the detectors are likewise created simultaneously. After a suitable statistical accuracy has been obtained, and the measurement is thus complete, one takes the statistics given by, say, detector 1, computes its moments α_k , and calculates the numbers β_k using (3). The β_k are now the moments of the distribution of Q , in the input state of the signal field. A similar procedure applied to detector 2 yields the moments β'_k of the distribution of P . One can then check whether (4) holds for both the sequences β_k and β'_k . If that is the case, then it is justified to say that (the statistics of) the position- and momentum-like observables Q and P have been measured together.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF YORK, YORK, UK
E-mail address: `pb516@york.ac.uk`

DEPARTMENT OF PHYSICS, UNIVERSITY OF TURKU, TURKU, FINLAND
E-mail address: `jukka.kiukas@utu.fi`

DEPARTMENT OF PHYSICS, UNIVERSITY OF TURKU, TURKU, FINLAND
E-mail address: `pekka.lahti@utu.fi`